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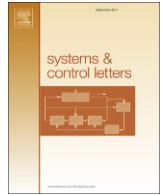
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# Regular implementation in the space of compactly supported functions

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## ABSTRACT

This article extends results on regular implementability in [P. Rocha, Canonical controllers and regular implementation of nd behaviors, in: Proceedings of the 16th IFAC World Congress, 2005] and [H.L. Trentelman, D. Napp Avelli, On the regular implementability of nD systems, Systems Control Lett. 56 (4) (2007) 265–271] to the case when the signal space is not an injective cogenerator, for instance, the space  $\mathcal{D}$  of compactly supported smooth functions on  $\mathbb{R}^n$ . In this case the bijective correspondence between behaviors and modules fails to hold; also projections and sums of behaviors need not in general be behaviors. A more general version of implementability is introduced and necessary and sufficient conditions are established for the implementation and regular implementation of a given desired behavior.

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## 1. Introduction

In this article we consider the problem of regular implementability over the space  $\mathcal{D}$  of compactly supported functions. Earlier work had treated this problem, both for lumped as well as distributed systems, but only when behaviors were considered in injective cogenerators, for example the space  $\mathcal{D}'$  of distributions on  $\mathbb{R}^n$  or the space  $\mathcal{C}^\infty$  of smooth functions [1,4,8,10,13]. The proofs there relied strongly on several facts – that the projection of a behavior is a behavior or that the sum of two behaviors is also one, for instance – facts that are not longer true when the space, such as  $\mathcal{D}$ , is not injective. Also important was the use of the categorical duality between behaviors and finitely generated modules [2, 12] by which questions about the former could be faithfully carried over to questions about the latter. This translation, which is a consequence of the cogenerator property of  $\mathcal{D}'$  and  $\mathcal{C}^\infty$  has to be modified even for the space of  $\mathcal{S}'$  of temperate distributions, which although injective does not cogenerate ([7, p. 171]).

In the spaces  $\mathcal{D}$ ,  $\mathcal{S}'$  (compactly supported distributions) and  $\mathcal{S}$  (the Schwartz space of rapidly decreasing functions), when the projection of a behavior may fail to be one [7], the question of implementability itself has to be reinterpreted. This paper

shows that a natural weakening of the question does admit a solution, in fact the same solution as when the space is an injective cogenerator.

To overcome the problem arising from the loss of Oberst's duality, this paper must rely on a PDE analogue of the Hilbert Nullstellensatz [5]. This statement is in terms of associated primes when the space is  $\mathcal{S}'$ , and is identical for the spaces  $\mathcal{D}$ ,  $\mathcal{S}'$  and  $\mathcal{S}$  (we collectively call these six spaces  $\mathcal{D}$ ,  $\mathcal{C}^\infty$ ,  $\mathcal{S}'$ ,  $\mathcal{S}$ ,  $\mathcal{S}'$  and  $\mathcal{D}$ , the *classical spaces*). Thus for ease of exposition we write specifically for the space  $\mathcal{D}$ . All results carry over, a fortiori, to the spaces  $\mathcal{S}'$  and  $\mathcal{S}$ , while we confine ourselves to a few remarks about  $\mathcal{S}'$ .

The paper is organized as follows – after recollecting the standard definition of regular implementability, we point out through examples the problems that now have to be overcome. This suggests a natural reformulation of the problem. The final choice of the controller equations (that will implement the controller) has to be chosen a little carefully – this requires the Nullstellensatz statement, here for  $\mathcal{D}$ . The last section is devoted to the construction of the implementing controller.

A word about the spaces  $\mathcal{D}$ ,  $\mathcal{S}'$  and  $\mathcal{S}$  in which we study the problem of (regular) implementability: It is standard practice in the theory of partial differential equations to estimate the growth of solutions at infinity. Our intention here is to carry over such questions, in its simplest setting, to the implementability problem. The growth condition that we impose in this paper is to require all signals to have decayed to zero at infinity. Compactly supported behaviors are also important for many other reasons. When they are dense in a  $\mathcal{D}$  or  $\mathcal{C}^\infty$  behavior – this happens precisely when

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the behavior is controllable – they can be used to approximate an arbitrary element of the behavior. Thus for controllable behaviors, the compactly supported sub-behavior already contains much of the information about the entire behavior.

*Remark on notation:* In [8]  $\mathcal{D}$  denotes the ring of constant coefficient differential operators and  $\mathcal{A}$  the signal space. Here, following L. Schwartz, we reserve  $\mathcal{D}$  for the space of test functions (i.e. compactly supported smooth functions) and use  $\mathcal{A}$  for the ring of differential operators. This is consistent with the notation in [5–7].

## 2. Preliminaries

The problem of regular implementability considered in [8,3] is solved there in two stages, first implementability and then the question of regularity. Here we follow the same pattern.

Let  $\mathcal{A} = \mathbb{C}[\partial_1, \dots, \partial_n]$  be the ring of constant coefficient partial differential operators on  $\mathbb{R}^n$ . Let  $\mathcal{D}$  be the space of distributions on  $\mathbb{R}^n$  considered as an  $\mathcal{A}$ -module, the module structure given by differentiation. Let  $\mathcal{F}$  be an  $\mathcal{A}$ -submodule of  $\mathcal{D}$ , here one of the six classical spaces listed above. Let  $m(\partial) = (m_1(\partial), \dots, m_k(\partial))$  be an element of  $\mathcal{A}^k$ . The behavior  $\mathfrak{B}_{\mathcal{F}}(m)$  of  $m(\partial)$  in  $\mathcal{F}$  is the kernel of the morphism

$$m(\partial) : \mathcal{F}^k \longrightarrow \mathcal{F}$$

$$f = (f_1, \dots, f_k) \longmapsto m(\partial)f = \sum_i m_i(\partial)f_i.$$

Let  $M$  be a submodule of  $\mathcal{A}^k$ . The behavior  $\mathfrak{B}_{\mathcal{F}}(M)$  of  $M$  in  $\mathcal{F}$  is  $\bigcap_{m \in M} \mathfrak{B}_{\mathcal{F}}(m)$ , the common behavior of all the  $m \in M$ . If  $(m_1, \dots, m_\ell)$ ,  $m_i = (m_{i1}(\partial), \dots, m_{ik}(\partial))$ , is any set of generators for  $M$ , then writing the entries  $m_{ij}(\partial)$  as a matrix  $M(\partial)$ , gives a morphism

$$M(\partial) : \mathcal{F}^k \longrightarrow \mathcal{F}^\ell$$

$$f \longmapsto M(\partial)f$$

whose kernel is the behavior  $\mathfrak{B}_{\mathcal{F}}(M)$  (we omit the subscript  $\mathcal{F}$  if it is clear from the context which space the behavior is located in).

Suppose there is given a natural splitting of the  $k$  coordinates of  $\mathcal{A}^k$ , and thus of  $\mathcal{F}^k$ , into two sets of  $r$  and  $s$  elements, i.e. say

$$\mathcal{A}^k \simeq \mathcal{A}^r \oplus \mathcal{A}^s$$

and the corresponding  $\mathcal{F}^k = \mathcal{F}^r \oplus \mathcal{F}^s$ . A submodule  $M_1$  of  $\mathcal{A}^r$  is identified naturally with a submodule of  $\mathcal{A}^k$  via the inclusion  $i_1 : \mathcal{A}^r \rightarrow \mathcal{A}^r \oplus \mathcal{A}^s, x \mapsto i_1(x) = (x, 0)$ , and similarly for submodules of  $\mathcal{A}^s$  via the inclusion  $i_2 : \mathcal{A}^s \rightarrow \mathcal{A}^r \oplus \mathcal{A}^s, y \mapsto i_2(y) = (0, y)$ . The submodule of  $\mathcal{A}^r \oplus \mathcal{A}^s$  consisting of all  $(x, 0)$  in  $M$  is denoted by  $\mathcal{A}^r \cap M$  and is isomorphic to  $i_1^{-1}(M)$ . Similarly,  $M \cap \mathcal{A}^s$  is the submodule  $\{(0, y) \in M\}$ , isomorphic to  $i_2^{-1}(M)$ . The behavior  $\mathfrak{B}_{\mathcal{F}}(M_1)$  in  $\mathcal{F}^r$  must now be identified with the behavior  $\mathfrak{B}_{\mathcal{F}}(M_1) \oplus \mathcal{F}^s$  in  $\mathcal{F}^r \oplus \mathcal{F}^s$  (as the behavior of the 0-submodule of  $\mathcal{A}^s$  is  $\mathcal{F}^s$ ), and similarly for the second factor.

Finally, let  $\pi_1$  and  $\pi_2$  denote the projections of  $\mathcal{F}^r \oplus \mathcal{F}^s$  onto the first and second factors.

*The problem of implementability:* Let  $\mathfrak{B}$  be a behavior in  $\mathcal{F}^r \oplus \mathcal{F}^s$  (the full plant behavior) and  $\mathcal{K}$  a behavior in  $\mathcal{F}^r$  (the manifest behavior that must be attained). Is there a behavior  $\mathcal{C}$  in  $\mathcal{F}^s$  (the controller behavior) such that

$$\pi_1(\mathfrak{B} \cap (\mathcal{F}^r \oplus \mathcal{C})) = \mathcal{K} \text{ ?}$$

If such controller behavior exists, then we call  $\mathcal{K}$  *implementable* with respect to  $\mathfrak{B}$ . We refer to the earlier work for the control theoretic significance of this problem, where it has been solved when  $\mathcal{F}$  is  $\mathcal{D}$  or  $\mathcal{C}^\infty$ , that is, for injective cogenerators [8].

Suppose now that  $\mathcal{F}$  is the space  $\mathcal{D}$  of compactly supported smooth functions. We then run into immediate difficulties, even with the above formulation of the problem, as now a projection of a behavior need not be a behavior [7].

**Example 1.** Let  $\mathcal{A} = \mathbb{C}[\frac{d}{dt}]$ , and let  $\pi_1 : \mathcal{D}^2 \rightarrow \mathcal{D}$  be the projection onto the first factor. Let  $M$  be the cyclic submodule of  $\mathcal{A}^2$  generated by  $(1, -\frac{d}{dt})$ . Then  $\mathfrak{B}_{\mathcal{D}}(M) = \{(\frac{df}{dt}, f) \mid f \in \mathcal{D}\}$ , but  $\pi_1(\mathfrak{B}_{\mathcal{D}}(M)) = \{\frac{df}{dt} \mid f \in \mathcal{D}\}$  is not a behavior in  $\mathcal{D}$ .

It is shown in [7] that even though a projection of a behavior need not be one, the smallest behavior containing it can be characterized. More precisely, the following is true.

**Proposition 2** ([7]). *Let  $\mathcal{F}$  be  $\mathcal{D}$ ,  $\mathcal{E}'$  or  $\mathcal{S}$ . Let  $M$  be a submodule of  $\mathcal{A}^k = \mathcal{A}^r \oplus \mathcal{A}^s$ . Then  $\mathfrak{B}_{\mathcal{F}}(i_1^{-1}(M)) \simeq \mathfrak{B}_{\mathcal{F}}(\mathcal{A}^r \cap M)$  is the smallest behavior containing  $\pi_1(\mathfrak{B}_{\mathcal{F}}(M))$ . If  $\mathcal{F}$  is an injective  $\mathcal{A}$ -module (such as  $\mathcal{D}$ ,  $\mathcal{C}^\infty$  or  $\mathcal{S}'$ ), then the projection of a behavior is also a behavior, so that these two behaviors are equal.*

**Example 1 (Continued).** The smallest behavior containing  $\pi_1(\mathfrak{B}_{\mathcal{D}}(M)) = \{\frac{df}{dt} \mid f \in \mathcal{D}\}$  is  $\mathcal{D}$  itself.

This suggests the following reformulation:

*Implementability in a classical space:* Let  $\mathcal{F}$  be a classical space, and  $\mathfrak{B}$  a behavior in  $\mathcal{F}^k = \mathcal{F}^r \oplus \mathcal{F}^s$ . Let  $\mathcal{K}$  be a (manifest) behavior in  $\mathcal{F}^r$ . Is there a controller behavior  $\mathcal{C}$  in  $\mathcal{F}^s$  such that

$\mathcal{K}$  is the smallest behavior containing  $\pi_1(\mathfrak{B} \cap (\mathcal{F}^r \oplus \mathcal{C}))$ ?

Such a controller  $\mathcal{C}$  is said to implement  $\mathcal{K}$  with respect to  $\mathfrak{B}$ . Again, if such a controller behavior exists, we call  $\mathcal{K}$  *implementable* with respect to  $\mathfrak{B}$ .

In the course of development of the results of this paper, we have to sometimes consider the sum of two behaviors. In general, the sum of two behaviors  $\mathfrak{B}(M_1)$  and  $\mathfrak{B}(M_2)$  need not be a behavior, but always, the smallest behavior containing this sum is  $\mathfrak{B}(M_1 \cap M_2)$ , see [6].

**Example 3.** Let  $\mathcal{A} = \mathbb{C}[\frac{d}{dt}]$  and let  $M_1$  and  $M_2$  be cyclic submodules of  $\mathcal{A}^2$  generated by  $(1, 0)$  and  $(1, -\frac{d}{dt})$  respectively. Then  $M_1 \cap M_2 = 0$ , so that  $\mathfrak{B}_{\mathcal{D}}(M_1 \cap M_2) = \mathcal{D}^2$ . On the other hand,  $\mathfrak{B}_{\mathcal{D}}(M_1) = \{(0, f) \mid f \in \mathcal{D}\}$  and  $\mathfrak{B}_{\mathcal{D}}(M_2) = \{(\frac{dg}{dt}, g) \mid g \in \mathcal{D}\}$ , and it is easy to see that  $\mathfrak{B}_{\mathcal{D}}(M_1) + \mathfrak{B}_{\mathcal{D}}(M_2)$  is not all  $\mathcal{D}^2$ . As  $\mathfrak{B}_{\mathcal{D}}(M_1 \cap M_2)$  is the smallest behavior containing this sum, it follows that this sum is not a behavior.

To say that the controller  $\mathcal{C}$  implements *regularly* is to say that the controller equations  $C$  have nothing in common with the behavior equations  $M$ , i.e.  $M \cap C = 0$ . This is really a statement about behaviors, namely that the smallest behavior containing  $\mathfrak{B} + \mathcal{C}$  is  $\mathcal{F}^k = \mathcal{F}^r \oplus \mathcal{F}^s$ , since 0 is the only submodule whose behavior is all of  $\mathcal{F}^k$ , in every classical space  $\mathcal{F}$ . Indeed, we have:

**Lemma 4.** *Let  $\mathcal{F}$  be a classical space. Let  $M, M'$  and  $C, C'$  be submodules of  $\mathcal{A}^k$  such that  $\mathfrak{B}_{\mathcal{F}}(M) = \mathfrak{B}_{\mathcal{F}}(M')$  and  $\mathfrak{B}_{\mathcal{F}}(C) = \mathfrak{B}_{\mathcal{F}}(C')$ . Then  $M \cap C = 0$  if and only if  $M' \cap C' = 0$*

**Proof.** For any classical space  $\mathcal{F}$ , the only submodule of  $\mathcal{A}^k$  whose behavior is all of  $\mathcal{F}^k$  is the 0-submodule. Thus if  $M \cap C = 0$ , then  $\mathcal{F}^k$  is the smallest behavior containing  $\mathfrak{B}_{\mathcal{F}}(M) + \mathfrak{B}_{\mathcal{F}}(C)$ , and hence also the smallest behavior containing  $\mathfrak{B}_{\mathcal{F}}(M') + \mathfrak{B}_{\mathcal{F}}(C')$ . This implies that  $M' \cap C' = 0$ .  $\square$

If, for a given  $\mathcal{K}$  there exists a controller behavior  $\mathcal{C}$  that implements  $\mathcal{K}$  regularly, then we call  $\mathcal{K}$  *regularly implementable* (with respect to  $\mathfrak{B}$ ).

A third difficulty encountered when the signal space is not an injective cogenerator is that there is no longer a bijective correspondence between behaviors in  $\mathcal{F}^k$  and submodules of  $\mathcal{A}^k$ ; indeed Lemma 4 is a statement about such a possibility.

**Example 5.** Let  $\mathcal{F} = \mathcal{D}$ , and  $M_1 = (\frac{d}{dt})$ ,  $M_2 = (1)$  ideals of  $\mathcal{A}$ . Then  $\mathfrak{B}_{\mathcal{D}}(M_1) = \mathfrak{B}_{\mathcal{D}}(M_2) = 0$ .

Thus, if  $M$  is a submodule of  $\mathcal{A}^k$  and  $\mathfrak{B}_{\mathcal{F}}(M)$  its behavior in  $\mathcal{F}^k$ , then the submodule of all elements  $m$  in  $\mathcal{A}^k$  such that  $\mathfrak{B}_{\mathcal{F}}(M) \subset \mathfrak{B}_{\mathcal{F}}(m)$  – which is in general larger than  $M$  – is called the (Willems) closure of  $M$  with respect to  $\mathcal{F}$ , and is denoted by  $\overline{M}_{\mathcal{F}}$  (we omit the subscript  $\mathcal{F}$  if it is clear from the context which space is considered). The calculation of this closure is analogous to the Hilbert Nullstellensatz, and is studied in [5] for the classical spaces. For the spaces  $\mathfrak{D}$ ,  $\mathcal{E}'$  and  $\mathcal{S}$ , this calculation is the following:

**Proposition 6** ([5]). *Let  $\mathcal{F}$  be  $\mathfrak{D}$ ,  $\mathcal{E}'$  or  $\mathcal{S}$ , and let  $M$  be a submodule of  $\mathcal{A}^k$ . Then the closure  $\overline{M}_{\mathcal{F}}$  of  $M$  with respect to these  $\mathcal{F}$  equals  $\{m \in \mathcal{A}^k \mid \exists a \in \mathcal{A}, a \neq 0 : am \in M\}$ . In other words, if  $\pi : \mathcal{A}^k \rightarrow \mathcal{A}^k/M$  is the natural projection, and  $\mathcal{T}$  is the torsion submodule of  $\mathcal{A}^k/M$ , then  $\overline{M}_{\mathcal{F}} = \pi^{-1}(\mathcal{T})$ .*

**Remark 7.** The papers [5,6] calculate the closure also with respect to the space  $\mathcal{S}'$  of temperate distributions. As this calculation is somewhat technical (and involves the associated primes of  $\mathcal{A}^k/M$ ) we do not include it here. Thus apart from some remark about  $\mathcal{S}'$ , we confine ourselves to the spaces  $\mathfrak{D}$ ,  $\mathcal{E}'$  and  $\mathcal{S}$ .

**Remark 8.** If  $M$  equals its closure  $\overline{M}_{\mathcal{F}}$ , then  $M$  is said to be *closed* with respect to  $\mathcal{F}$ . By definition the behavior of a submodule  $M$  equals that of its closure  $\overline{M}_{\mathcal{F}}$ ; indeed  $\overline{M}_{\mathcal{F}}$  is the largest submodule with the same behavior as that of  $M$ . Given a behavior  $\mathfrak{B}$  we denote this largest submodule by  $\mathcal{M}(\mathfrak{B})$  and call it the *vanishing module* of  $\mathfrak{B}$ . Thus  $\mathcal{M}(\mathfrak{B}_{\mathcal{F}}(M)) = \overline{M}_{\mathcal{F}}$ .

### 3. Implementability

The first result of this section is a characterization of those behaviors  $\mathcal{K}$  that are implementable when the underlying signal space is  $\mathfrak{D}$  (but, as noted above, the result is equally valid for  $\mathcal{E}'$  and  $\mathcal{S}$ ). In the remainder of this paper, for any given submodule  $M$  of  $\mathcal{A}^k$  we will write  $\mathfrak{B}(M)$  instead of  $\mathfrak{B}_{\mathfrak{D}}(M)$ .

**Theorem 9.** *Let  $\mathfrak{B}$  the full plant behavior in  $\mathfrak{D} \oplus \mathfrak{D}$ , and  $\mathcal{K}$  a behavior in  $\mathfrak{D}$  be given. Let  $M = \mathcal{M}(\mathfrak{B})$  be the vanishing module of the plant behavior. Then the following are equivalent:*

1.  $\mathcal{K}$  is implementable with respect to  $\mathfrak{B}$  i.e., there is a behavior  $\mathcal{C}$  in  $\mathfrak{D}$  such that  $\mathcal{K}$  is the smallest behavior containing  $\pi_1(\mathfrak{B} \cap (\mathfrak{D} \oplus \mathcal{C}))$  (so that if this projection is itself a behavior, then it equals  $\mathcal{K}$ ).
2.  $\mathfrak{B}(\pi_1(M)) \subset \mathcal{K} \subset \mathfrak{B}(\mathcal{A}^r \cap M)$ .
3. The submodule  $K := \mathcal{M}(\mathcal{K}) \cap \pi_1(M)$  satisfies  $\mathcal{K} = \mathfrak{B}(K)$  and  $\mathcal{A}^r \cap M \subset K$ .
4. There is a submodule  $K$  of  $\mathcal{A}^r$  such that  $\mathcal{K} = \mathfrak{B}(K)$  and  $\mathcal{A}^r \cap M \subset K \subset \pi_1(M)$ .

Furthermore, if any of the above statements hold, then for any submodule  $K$  of  $\mathcal{A}^r$  such that (4) holds, the canonical controller  $\mathfrak{B}(\pi_2(\tilde{M}(K)))$ , with  $\tilde{M}(K) = \{(m_1, m_2) \in M \mid (m_1, 0) \in K\}$  implements  $\mathcal{K}$  in the sense of (1).

**Proof.** (1) $\Rightarrow$ (2). As  $\mathcal{C}$  is a behavior in  $\mathfrak{D}$ , it must satisfy  $0 \subset \mathcal{C} \subset \mathfrak{D}$ . If  $\mathcal{C} = 0$ , then  $\mathfrak{B} \cap (\mathfrak{D} \oplus 0) = \{(w, 0) \in \mathfrak{B}\} \simeq i_1^{-1}(\mathfrak{B})$  where  $i_1 : \mathfrak{D} \rightarrow \mathfrak{D} \oplus \mathfrak{D}$  is the canonical inclusion. By Proposition 4.1 of [7] quoted earlier,  $i_1^{-1}(\mathfrak{B}) = \mathfrak{B}(\pi_1(M))$ .

Suppose now that  $\mathcal{C} = \mathfrak{D}$ . Then  $\mathfrak{B} \cap (\mathfrak{D} \oplus \mathfrak{D}) = \mathfrak{B}$ . The projection  $\pi_1(\mathfrak{B})$  may not be a behavior, but the smallest behavior containing it is  $\mathfrak{B}(i_1^{-1}(M))$  (by Proposition 4.2 of [7], also quoted earlier). By the identification of Section 2, this is precisely  $\mathfrak{B}(\mathcal{A}^r \cap M)$ .

(2) $\Rightarrow$ (3). Taking the vanishing module of the three behaviors in (2) gives

$$\mathcal{M}(\mathfrak{B}(\mathcal{A}^r \cap M)) \subset \mathcal{M}(\mathcal{K}) \subset \mathcal{M}(\mathfrak{B}(\pi_1(M))).$$

As  $M = \mathcal{M}(\mathfrak{B})$ ,  $M$  is closed (here with respect to  $\mathfrak{D}$ ), and hence so is  $\mathcal{A}^r \cap M$  ( $\mathcal{A}^r$  is of course closed, and the intersection of closed submodules is closed [6]). The above inclusions then become

$$\mathcal{A}^r \cap M \subset \mathcal{M}(\mathcal{K}) \subset \overline{\pi_1(M)}.$$

Let  $\mathcal{M}(\mathcal{K}) \cap \pi_1(M) = K$ . Then  $\mathfrak{B}(\mathcal{M}(\mathcal{K})) + \mathfrak{B}(\pi_1(M))\mathcal{K} + \mathfrak{B}(\pi_1(M)) \subset \mathfrak{B}(K)$  – in general this inclusion may be strict and only  $\mathfrak{B}(K)$  is the smallest behavior that contains this sum [6], but here, by (2),  $\mathfrak{B}(\pi_1(M)) \subset \mathcal{K}$ , and hence  $\mathcal{K} + \mathfrak{B}(\pi_1(M))$  equals  $\mathcal{K}$ . Then, as  $\mathfrak{B}(K)$  is the smallest behavior that contains  $\mathcal{K}$ , it must be equal to  $\mathcal{K}$ .

(3) implies (4) is obvious.

(4) $\Rightarrow$ (1). Given  $K$  satisfying (4), let  $\tilde{M}(K) = \{(m_1, m_2) \in M \mid (m_1, 0) \in K\}$  (our convention requires us to consider submodules of  $\mathcal{A}^r$  as submodules of  $\mathcal{A}^r \oplus \mathcal{A}^s$ ). Let  $C_{can} = \pi_2(\tilde{M}(K))$  – this is the canonical controller associated to  $K$ , see [8]. Interconnecting the plant to the controller defined by  $C_{can}$  is to augment the laws of  $M$  by the laws of  $C_{can}$ ; This yields the module  $M + C_{can}$ . The submodule  $\mathcal{A}^r \cap (M + C_{can})$  equals  $K$ , by the very construction of the canonical controller.

Let  $\mathcal{C} \subset \mathfrak{D}$  be the behavior of  $C_{can}$  considered as a submodule of  $\mathcal{A}^s$ . Identifying  $C_{can}$  with a submodule of  $\mathcal{A}^r \oplus \mathcal{A}^s$  identifies  $\mathcal{C}$  with the behavior  $\mathfrak{D} \oplus \mathcal{C}$  in  $\mathfrak{D} \oplus \mathfrak{D}$ . The behavior of  $M + C_{can}$  is then  $\mathfrak{B}(M) \cap \mathfrak{B}(C_{can}) = \mathfrak{B} \cap (\mathfrak{D} \oplus \mathcal{C})$ . The projection  $\pi_1(\mathfrak{B} \cap (\mathfrak{D} \oplus \mathcal{C}))$  is in general not a behavior, but the smallest behavior containing it is the behavior of  $\mathcal{A}^r \cap (M + C_{can}) = K$ , which equals  $\mathcal{K}$ .  $\square$

**Remark 10.** Note that condition 2 of Theorem 9 differs from the usual characterization of implementability that states that the desired behavior  $\mathcal{K}$  should be wedged in between the hidden behavior and the manifest plant behavior (see [11]). Although  $\mathfrak{B}(\pi_1(M))$  is indeed the hidden behavior,  $\mathfrak{B}(\mathcal{A}^r \cap M)$  is only the smallest behavior that contains the projection  $\pi_1(\mathfrak{B})$ . This projection, in general, need not be a behavior.

**Remark 11.** It is an easy check that the canonical controller  $\pi_2(\tilde{M}(K))$  is equal to  $(K + M) \cap \mathcal{A}^s$  [8,9].

**Remark 12.** From the proof of the above theorem it is clear that in order to construct, even define, the canonical controller or any other implementing controller, it is necessary that a submodule defining the behavior  $\mathcal{K}$  be contained in  $\pi_1(M)$ . A priori, this need not be so, and the above necessary condition must be forced as in the proof of the theorem. The next example illustrates this problem.

**Example 13.** Let  $\mathcal{A} = \mathbb{C}[\frac{d}{dt}]$ ,  $\mathfrak{B} = \mathfrak{B}_{\mathfrak{D}}(M) \subset \mathfrak{D}^3 \oplus \mathfrak{D}$  and  $\mathcal{K} = \mathfrak{B}_{\mathfrak{D}}(K) \subset \mathfrak{D}^3$  where

$$M = \begin{pmatrix} 0 & \frac{d^2}{dt^2} - \frac{d}{dt} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & \frac{d}{dt} - 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then  $\mathcal{A}^3 \cap M \subset K \subset \overline{\pi_1(M)}$  but  $K$  is not contained in  $\pi_1(M)$ .

The above theorem solves the implementation problem by *partial* interconnection, where the term partial refers to the fact that the controller equations must be a submodule of  $\mathcal{A}^s \subset \mathcal{A}^r \oplus \mathcal{A}^s$ . If the controller equations are not restricted in this way, i.e. if they could be an arbitrary submodule of  $\mathcal{A}^r \oplus \mathcal{A}^s$ , then the interconnection of such a controller is said to be *full*. To be more precise, if  $\mathfrak{B} \subset \mathfrak{D}^k$ , and  $\mathcal{K}$  is a given behavior in  $\mathfrak{D}^k$ , then we call  $\mathcal{K}$  *implementable by full interconnection* with respect to  $\mathfrak{B}$  if there exists a controller behavior  $\mathcal{C} \subset \mathfrak{D}^k$  such that  $\mathfrak{B} \cap \mathcal{C} = \mathcal{K}$ . Note that in the case of full interconnection the definition of implementability is the same for all classical spaces, since the intersection of two behaviors is



always a behavior. The given behavior  $\mathcal{K}$  is said to be *regularly implementable by full interconnection* if there exists a controller behavior  $\mathcal{C}$  such that  $\mathfrak{B} \cap \mathcal{C} = \mathcal{K}$  and  $\mathfrak{B} + \mathcal{C} = \mathfrak{D}^k$ .

Full interconnection problems are in general easier to solve. In Rocha [3], the problem of implementability by partial interconnection was converted to a full interconnection implementability problem (see also [8], Cor. 14). We now extend this result to the case when the signal space is  $\mathfrak{D}$  (or  $\mathcal{E}'$  or  $\mathcal{J}$ ). Again, as  $\mathfrak{D}$  is not an injective cogenerator, its proof requires more care.

**Theorem 14.** *Let  $\mathfrak{B}$  and  $\mathcal{K}$  be given as in Theorem 9. Let  $M = \mathcal{M}(\mathfrak{B})$  be the vanishing module of  $\mathfrak{B}$ . Assume that statement (1) of Theorem 9 holds and choose any  $K$  such that (4) of Theorem 9 holds. Let  $\tilde{M}(K) = \{(m_1, m_2) \in M \mid (m_1, 0) \in K\}$ , and let  $\mathfrak{B}(\pi_2(\tilde{M}(K)))$  be the canonical controller. Let  $\mathcal{C}$  be a behavior in  $\mathfrak{D}$ , and let  $C = \mathcal{M}(\mathcal{C})$  be its vanishing module. Then we have the following:*

1. *The controller behavior  $\mathcal{C}$  implements  $\mathcal{K}$  with respect to  $\mathfrak{B}$  if and only if  $\mathfrak{B}(C \cap \pi_2(M))$  implements  $\mathfrak{B}(\pi_2(\tilde{M}(K)))$  by full interconnection with respect to  $\mathfrak{B}(M \cap \mathcal{A}^s)$ .*
2. *If  $\mathcal{C}$  implements  $\mathfrak{B}(\pi_2(\tilde{M}(K)))$  by full interconnection with respect to  $\mathfrak{B}(M \cap \mathcal{A}^s)$  then  $\mathcal{C}$  also implements  $\mathcal{K}$  with respect to  $\mathfrak{B}$  by partial interconnection.*

Furthermore, (1) and (2) also hold with ‘implements’ replaced by ‘regularly implements’.

In order to prove this theorem, we need a few lemmas. Our first lemma states that for any submodule  $K$  of  $\mathcal{A}^r$  the canonical controller associated with  $K$  and its closure  $\bar{K}$  coincide.

**Lemma 15.** *Let  $K \subset \mathcal{A}^r$ . Then  $\mathfrak{B}(\pi_2(\tilde{M}(K))) = \mathfrak{B}(\pi_2(\tilde{M}(\bar{K})))$ .*

**Proof.** The inclusion  $\supset$  follows from the fact that  $\tilde{M}(K) \subset \tilde{M}(\bar{K})$ . Conversely, we claim that

$$\pi_2(\tilde{M}(\bar{K})) \subset \pi_2(\tilde{M}(K)).$$

Indeed, let  $(0, m_2) \in \pi_2(\tilde{M}(\bar{K}))$ . Then there exists  $m_1$  such that  $(m_1, m_2) \in \tilde{M}(\bar{K})$ . Thus  $(m_1, m_2) \in M$  and  $(m_1, 0) \in \bar{K}$ . There exists  $0 \neq a \in \mathcal{A}$  such that  $(am_1, 0) \in K$ . This implies that  $(am_1, am_2) \in \tilde{M}(K)$ , which yields  $(0, am_2) \in \pi_2(\tilde{M}(K))$ . We conclude that  $(0, m_2) \in \pi_2(\tilde{M}(K))$  as claimed.  $\square$

Using this lemma, we now prove that the canonical controller is in fact uniquely determined by the behavior  $\mathcal{K}$  and not by its representing submodule  $K$ :

**Lemma 16.** *Let  $K_1, K_2 \subset \mathcal{A}^r$  such that  $\mathfrak{B}(K_1) = \mathfrak{B}(K_2)$ . Then  $\mathfrak{B}(\pi_2(\tilde{M}(K_1))) = \mathfrak{B}(\pi_2(\tilde{M}(K_2)))$ .*

**Proof.**  $\mathfrak{B}(K_1) = \mathfrak{B}(K_2)$  implies  $\bar{K}_1 = \bar{K}_2$ . The result then follows from the previous lemma.  $\square$

The next lemma was proven in [8]:

**Lemma 17.** *Let  $M \subset \mathcal{A}^r \oplus \mathcal{A}^s$  and  $C \subset \mathcal{A}^s$ . Suppose that  $K$  is a submodule of  $\mathcal{A}^r$  such that  $\mathcal{A}^r \cap M \subset K \subset \pi_1(M)$ . Then the following are equivalent:*

- (a)  $\mathcal{A}^r \cap (M + C) = K$ .
- (b)  $(M + (C \cap \pi_2(M))) \cap \mathcal{A}^s = \pi_2(\tilde{M}(K))$ .

Further,  $M \cap C = 0$  if and only if  $M \cap (C \cap \pi_2(M)) = 0$ .

**Proof.** This is Lemma 13 in [8].  $\square$

Finally, we need the following technical result:

**Lemma 18.** *With the assumptions of Lemma 17 the following hold:*

1.  $(M + C) \cap \mathcal{A}^s = \pi_2(\tilde{M}(K))$  implies  $\mathcal{A}^r \cap (M + C) = \bar{K}$ .
2.  $(M + (C \cap \pi_2(M))) \cap \mathcal{A}^s = \pi_2(\tilde{M}(K))$  implies  $\mathcal{A}^r \cap (M + C) = \bar{K}$ .

**Proof.** (1): Let  $x$  be in  $\bar{K}$ . Then for some nonzero  $a$  in  $\mathcal{A}$ ,  $ax(m_1, 0)$  is in  $K$ . As  $K$  is contained in  $\pi_1(M)$ , there is an  $m_2$  such that  $(m_1, m_2)$  is in  $\tilde{M}(K)$ , so that  $(0, m_2)$  is in  $\pi_2(\tilde{M}(K))$ . Now the assumption  $\pi_2(\tilde{M}(K)) = (M + C) \cap \mathcal{A}^s$  implies that there is a nonzero  $b$  in  $\mathcal{A}$  such that  $(0, bm_2)$  equals  $(0, m'_2) + (0, c)$ , where  $(0, c)$  is in  $C$  and  $(0, m'_2)$  is in  $M$ . Consider now  $abx = (bm_1, 0)$  in  $K$ . By the above,  $(bm_1, 0) = (bm_1, bm_2) - (0, bm_2) = (bm_1, bm_2) - (0, c) - (0, m'_2)$  is in  $M + C$ , and hence in  $\mathcal{A}^s \cap (M + C)$ . As  $\mathcal{A}$  is an integral domain,  $ab \neq 0$ . Hence  $x$  is in  $\mathcal{A}^s \cap (M + C) = \mathcal{A}^s \cap (M + C)$  (the closure of a finite intersection is the intersection of the closures [6]).

Conversely, suppose that  $x$  is in  $\mathcal{A}^r \cap (M + C)$ . Then for some nonzero  $a$ ,  $ax$  is in  $\mathcal{A}^r \cap (M + C)$ . Thus  $ax$  is of the form  $(m, c) + (0, -c)$ , where  $(m, c)$  is in  $M$  and  $(0, -c)$  is in  $C$ . This  $(0, -c)$  is also in  $(M + C) \cap \mathcal{A}^s$ , which implies that  $(0, -c)$  is in  $\pi_2(\tilde{M}(K))$  by the assumption of the lemma. Again, by definition of the closure with respect to  $\mathfrak{D}$ , there is a nonzero  $b$  in  $\mathcal{A}$  such that  $(0, -bc)$  is in  $\pi_2(\tilde{M}(K))$ . By Remark 11,  $\pi_2(\tilde{M}(K))$  equals  $(K + M) \cap \mathcal{A}^s$ . Thus it follows that  $(0, -bc) = (-m_1, 0) + (m_1, -bc)$  with  $(-m_1, 0)$  in  $K$  and  $(m_1, -bc)$  in  $M$ . This then implies that  $abx = (bm, bc) + (0, -bc) = (bm, bc) + (m_1, -bc) + (-m_1, 0) = (bm + m_1, 0) + (-m_1, 0)$  where  $(bm + m_1, 0)$  is in  $\mathcal{A}^r \cap M$  and hence by the assumption (of Lemma 17) on  $K$ . Thus  $abx$  is in  $K$ , and again as  $ab \neq 0$ , this implies that  $x$  is in  $\bar{K}$ .

(2): Assume  $(M + (C \cap \pi_2(M))) \cap \mathcal{A}^s = \pi_2(\tilde{M}(K))$ . Then by applying part (1) of this theorem one has that  $\bar{K} = \mathcal{A}^r \cap (M + (C \cap \pi_2(M))) \subset \mathcal{A}^r \cap (M + C)$ .

Conversely, suppose that  $x$  is in  $\mathcal{A}^r \cap (M + C)$ . Then for some nonzero  $a$ ,  $ax$  is in  $\mathcal{A}^r \cap (M + C)$ . Thus  $ax$  is of the form  $(m, c) + (0, -c)$ , where  $(m, c)$  is in  $M$  and  $(0, -c)$  is in  $C$ . Moreover,  $(0, -c)$  is in  $C \cap \pi_2(M)$ . Then this  $(0, -c)$  is also in  $(M + (C \cap \pi_2(M))) \cap \mathcal{A}^s$ , which implies that  $(0, -c)$  is in  $\pi_2(\tilde{M}(K))$  by the assumption. Again, by definition of the closure with respect to  $\mathfrak{D}$ , there is a nonzero  $b$  in  $\mathcal{A}$  such that  $(0, -bc)$  is in  $\pi_2(\tilde{M}(K))$ . By Remark 11,  $\pi_2(\tilde{M}(K))$  equals  $(K + M) \cap \mathcal{A}^s$ . Thus it follows that  $(0, -bc) = (-m_1, 0) + (m_1, -bc)$  with  $(-m_1, 0)$  in  $K$  and  $(m_1, -bc)$  in  $M$ . This then implies that  $abx = (bm, bc) + (0, -bc) = (bm, bc) + (m_1, -bc) + (-m_1, 0) = (bm + m_1, 0) + (-m_1, 0)$  where  $(bm + m_1, 0)$  is in  $\mathcal{A}^r \cap M$ , and hence by the assumption of Lemma 17, in  $K$ . Thus  $abx$  is in  $K$ , and again as  $ab \neq 0$ , this implies that  $x$  is in  $\bar{K}$ .  $\square$

We are now ready to give a proof of Theorem 14:

**Proof.** (1)  $(\Rightarrow)$ : Let  $K' = \mathcal{A}^r \cap (M + C)$ . Then  $\mathfrak{B}(K')$ , being the smallest behavior containing  $\pi_1(\mathfrak{B} \cap (\mathfrak{D} \oplus \mathfrak{B}(C)))$ , equals  $\mathcal{K}$ . Clearly  $\mathcal{A}^r \cap M \subset \mathcal{A}^r \cap (M + C) \subset \pi_1(M)$ , so that  $K'$  satisfies the assumption of Lemma 17. Then  $K'$  also satisfies (b) of Lemma 17. Let  $C' = C \cap \pi_2(M)$ . The behavior  $\mathfrak{B}(M + C') \cap \mathcal{A}^s = \mathfrak{B}(M \cap \mathcal{A}^s) + C' = \mathfrak{B}(M \cap \mathcal{A}^s) \cap \mathfrak{B}(C')$  thus equals  $\mathfrak{B}(\pi_2(\tilde{M}(K))) = \mathfrak{B}(\pi_2(\tilde{M}(K)))$  by Lemma 16.

$(\Leftarrow)$ : Assume that  $\mathfrak{B}(C \cap \pi_2(M))$  implements  $\mathfrak{B}(\pi_2(\tilde{M}(K)))$  by full interconnection with respect to  $\mathfrak{B}(M \cap \mathcal{A}^s)$ , that is  $\mathfrak{B}(M \cap \mathcal{A}^s) \cap \mathfrak{B}(C \cap \pi_2(M)) = \mathfrak{B}(\pi_2(\tilde{M}(K)))$ . Then  $\mathfrak{B}(M \cap \mathcal{A}^s) + (C \cap \pi_2(M)) = \mathfrak{B}(\pi_2(\tilde{M}(K)))$ , so that taking the vanishing module of both these behaviors gives

$$(M \cap \mathcal{A}^s) + (C \cap \pi_2(M)) = (M + (C \cap \pi_2(M))) \cap \mathcal{A}^s = \pi_2(\tilde{M}(K))$$

(where the first equality is the ‘modular law’ – the collection of submodules of  $\mathcal{A}^s$  is a modular lattice). By Lemma 18 part (2), it follows that  $\mathcal{A}^r \cap (M + C) = \bar{K}$ . The behavior of  $\mathcal{A}^r \cap (M + C)$  equals that of  $\mathcal{A}^r \cap (M + C)$  – by the definition of closure – and this is the smallest behavior containing  $\pi_1(\mathfrak{B} \cap (\mathfrak{D} \oplus \mathfrak{B}(C)))$  [7].

(2) Assume  $\mathfrak{B}(C)$  implements  $\mathfrak{B}(\pi_2(\tilde{M}(K)))$  by full interconnection with respect to  $\mathfrak{B}(M \cap \mathcal{A}^s)$ , that is  $\mathfrak{B}(M \cap \mathcal{A}^s) \cap \mathfrak{B}(C) = \mathfrak{B}(\pi_2(\tilde{M}(K)))$ . Then it follows that  $\mathfrak{B}(M \cap \mathcal{A}^s) + C = \mathfrak{B}(M \cap \mathcal{A}^s) \cap \mathfrak{B}(C) = \mathfrak{B}(\pi_2(\tilde{M}(K)))$ , so that taking the vanishing module of both these behaviors gives

$$(M \cap \mathcal{A}^s) + C = (M + C) \cap \mathcal{A}^s = \pi_2(\tilde{M}(K))$$

(where the first equality is the modular law). By Lemma 18, it follows that  $\mathcal{A}' \cap (\overline{M + C}) = \overline{K}$ . The behavior of  $\mathcal{A}' \cap (\overline{M + C})$  equals that of  $\mathcal{A}' \cap (M + C)$  – by the definition of closure – and this is the smallest behavior containing  $\pi_1(\mathfrak{B} \cap (\mathfrak{D} \oplus \mathfrak{B}(C)))$  [7]. This establishes (1).

Finally, both statements (1) and (2) hold with ‘implements’ replaced by ‘regularly implements’. For statement (1) this follows from the fact that for  $C \subset \mathcal{A}^s$  we have  $M \cap C = 0$  if and only if  $M \cap (C \cap \pi_2(M)) = 0$  (see Lemma 17). For statement (2) we obviously have  $M \cap C = 0$  if and only if  $(M \cap \mathcal{A}^s) \cap C = 0$ .  $\square$

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